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ON THE FREE COOLING OF A HOMOGENEOUS SPHERE, OF INITIAL UNIFORM TEMPERATURE, IN A MEDIUM WHICH MAINTAINS A CONSTANT SURFACE TEMPERATURE.

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1. The problem of the cooling of a homogeneous sphere initially heated to a uniform temperature divides itself naturally into two cases, namely: first, that in which the dissipation of heat at the surface of the sphere goes on independently of the surrounding medium, and second, that in which the dissipation is conditioned by the surrounding medium. For brevity we have applied the phrase "free cooling" to designate the first case. Both cases have been stated by Fourier and subsequent writers, but their investigations are, so far as we know, confined almost wholly to the second and more complex case. Moreover, they have, for the most part, contemplated the subject from the standpoint of the pure analyst; they have been too much occupied with the difficulties of their splendid analysis to give much heed to the needs of the computer.

The most interesting example for application of the theory of a cooling sphere is presented by the earth. The hypothesis most commonly entertained is, that the whole mass of the earth was at a certain epoch heated to a uniform temperature, and is now slowly cooling by conduction without sensibly heating surrounding space. With the ultimate object in view of tracing out the consequences of this hypothesis, we have sought to express the solutions of the case of free cooling and that of conditioned cooling in such terms that the computer can, without undue labor, assign the temperature at any point within the sphere for any value of the time.

In the following pages the case of the free cooling subject to the restriction of a constant coefficient of diffusion is alone considered. How closely the conditions of this restricted case accord with those actually presented by the earth is a question which requires examination in its proper place. The present enquiry, however, is directed solely to the mathematical treatment of the problem, and the conditions are tacitly assumed to apply to the earth.

2. The law of cooling of a homogeneous sphere, initially heated to a uniform temperature and cooling in a medium which maintains a constant surface temperature, must evidently be such as to give the same temperature for all points equidistant from the centre of the sphere.

Let u_0 be the initial uniform excess of the temperature of the sphere over that of the surrounding medium. Let u be the excess of the temperature of any shell of radius r and thickness dr over the temperature of the surrounding medium at any time t, after the initial epoch. Denote the radius of the external surface of the sphere by r_0; and let the coefficient of diffusion which is here assumed to be constant, be symbolized by a^2. Then the function of u_0, u, r_0, r, t, and a^2 which expresses the law of cooling must be such as to satisfy the following partial differential equation: —

$$\frac{\partial (ru)}{\partial t} = a^2 \frac{\partial^2 (ru)}{\partial r^2}\,. \tag{1}$$

If we write the functional relation which is to be discovered in the form

$$ru = f(u_0, r_0, r, t, a^2),$$

we must also have

$$ru = 0 \quad \text{for} \quad r = r_0, \tag{2}$$

$$ru = 0 \quad \text{for} \quad r = 0, \tag{3}$$

$$ru = ru_0 \quad \text{for} \quad t = 0, \tag{4}$$

$$ru = 0 \quad \text{for} \quad t = \infty\,. \tag{5}$$

Without going through the steps essential to exclude other functions, we observe that conditions (1) and (2) are satisfied by the expression

$$C_n e^{-a^2 (n\pi / r_0)^2 t} \sin n\pi r / r_0,$$

C_n being any constant, and n any integer. Hence, since every term of this form will satisfy (1) and (2), we may write

$$ru = \sum_{n=1}^{n=\infty} C_n e^{-a^2 (n\pi / r_0)^2 t} \sin n\pi r / r_0. \tag{6}$$

When $t = 0$, (6) becomes in accordance with (4),

$$ru_0 = \sum_{n=1}^{n=\infty} C_n \sin n\pi r / r_0. \tag{7}$$

To fulfill this condition we must have

$$u_0 \int_0^{r_0} r \sin n\pi \frac{r}{r_0}\, dr = C_n \int_0^{r_0} \sin^2 n\pi \frac{r}{r_0}\, dr. \tag{8}$$

If we put

$$\theta = \pi \frac{r}{r_0},$$

(8) becomes

$$\frac{r_0 u_0}{\pi} \int_0^{\pi} \theta \sin n\theta \, d\theta = C_n \int_0^{\pi} \sin^2 n\theta \, d\theta,$$

whence

$$C_n = -\frac{2r_0 u_0}{\pi} \cdot \frac{\cos n\pi}{n}$$

$$= \frac{2r_0 u_0}{\pi} \cdot \frac{(-1)^{n+1}}{n}\,. \tag{9}$$

Substituting this value of C_n in (6), we find the following equation, which expresses the law of cooling sought, in its simplest form: —

$$
\begin{aligned}
ru &= \frac{2r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} e^{-a^2 (n\pi / r_0)^2 t} \sin n\pi \frac{r}{r_0} \\
&= \frac{2r_0 u_0}{\pi} \left\{ \begin{array}{l} + e^{-a^2 (\pi / r_0)^2 t} \sin \pi \frac{r}{r_0} \\ - \frac{1}{2} e^{-a^2 (2\pi / r_0)^2 t} \sin 2\pi \frac{r}{r_0} \\ + \frac{1}{3} e^{-a^2 (3\pi / r_0)^2 t} \sin 3\pi \frac{r}{r_0} \\ - \ldots\ldots\ldots\ldots \end{array} \right\}.
\end{aligned}
\qquad (10)
$$

The equation (10) satisfies all of the conditions (1) to (5), because each term satisfies them.

When $r = 0$, (10) gives for the temperature at the centre of the sphere,

$$u = \frac{2r_0 u_0}{\pi} \cdot \frac{0}{0} = \frac{2r_0 u_0}{\pi} \cdot \frac{d(ru)}{dr}.$$

Evaluating this expression, we find for the temperature at the centre of the sphere,

$$\underset{r=0.}{u} = 2u_0 \left[e^{-a^2 (\pi / r_0)^2 t} - e^{-a^2 (2\pi / r_0)^2 t} + e^{-a^2 (3\pi / r_0)^2 t} - \ldots \right]. \qquad (11)$$

When $t = 0$ this gives $u = u_0$, as may be readily seen by recurring to (10). When $t > 0$ it appears that $u < u_0$.

3. The series (10) and (11) express all the circumstances of the cooling sphere under the assumed conditions. They are rapidly converging series when $a^2(\pi / r_0)^2 t$ is not much less than unity, since the negative exponents in the successive terms increase as the squares of the natural numbers. But in the most important application of this theory, namely, to the earth, the value of $a^2(\pi / r_0)^2 t$ is very small unless the time, t, is very large. Thus, for the earth, if the British foot and the year are the space and time units, we have in round numbers

$$a^2 = 400,^*$$

$$r_0 = 21\,000\,000 \text{ feet.}$$

Hence, in order to make $a^2(\pi / r_0)^2 t$ unity, we must have

$$t = \frac{(21\,000\,000)^2}{400\pi^2},$$

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[*] See section (*o*) of Sir William Thompson's paper "On the Secular Cooling of the Earth." Thompson and Tait's Natural Philosophy, Vol. I. Part II. Appendix (D). The coefficient refers to the unit of bulk of the substance of the earth's crust.

or about 100,000,000,000 years. When applied to the case of the earth, therefore, the series (10) and (11) are too slowly converging for the purposes of computation except when the time is very great.

To overcome this practical difficulty and obtain an expression which will readily give the temperature u, of any point of the sphere at any time, we proceed to transform the series (10.)

4. In the transformation which follows, and again in the sequel, we shall make use of a certain Eulerian integral or Gamma function whose value may here be briefly demonstrated, the reader being referred for a fuller investigation to the better treatises on the integral calculus. The integral is

$$v = \int_0^\infty e^{-a^2x^2} dx \cos \beta x = \frac{\sqrt{\pi}}{2a} e^{-(\frac{1}{2}\beta / a)^2}.$$

To prove the equality of the second and third members of this equation, differentiate with respect to β and then integrate by parts. Thus

$$\frac{\partial v}{\partial \beta} = -\int_0^\infty e^{-a^2x^2} x dx \sin \beta x$$

$$= \left[\frac{e^{-a^2x^2} \sin \beta x}{2a^2}\right]_0^\infty - \frac{\beta}{2a^2}\int_0^\infty e^{-a^2x^2} dx \cos \beta x.$$

That is

$$\frac{\partial v}{\partial \beta} = -\frac{v\beta}{2a^2},$$

or

$$\frac{\partial v}{v} = -\frac{\beta \partial \beta}{2a^2};$$

whence

$$\log v = -\frac{\beta^2}{4a^2} + \log v_0,$$

or

$$v = v_0 e^{-(\frac{1}{2}\beta / a)^2}.$$

To determine the constant v_0 we observe that when $\beta = 0$,

$$v = v_0 = \int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}.$$

Therefore

$$v = \frac{\sqrt{\pi}}{2a} e^{-(\frac{1}{2}\beta / a)^2} \quad \text{and} \quad e^{-(\frac{1}{2}\beta / a)^2} = \frac{2a}{\sqrt{\pi}}\int_0^\infty e^{-a^2x^2} dx \cos \beta x.$$

5. Applying the preceding integral to the n^{th} term of equation (10), putting

$$\beta^2 = n^2\pi^2 \quad \text{and} \quad 4\alpha^2 = r_0^2/(a^2t),$$

we have

$$e^{-a^2(n\pi/r_0)^2t} = e^{-n^2\pi^2/(r_0^2/a^2t)} = \frac{r_0}{a\sqrt{(\pi t)}}\int_0^\infty e^{-r_0^2x^2/(4a^2t)}\,dx\cos n\pi x.$$

Hence equation (10) may be written

$$ru = \frac{2r_0^2u_0}{a\pi\sqrt{(\pi t)}}\int_0^\infty e^{-r_0^2x^2(4a^2t)^{-1}}\,dx\sum_{n=1}^{n=\infty}(-1)^{n+1}n^{-1}\cos nx\sin n\pi\frac{r}{r_0}. \quad (12)$$

Now, for brevity, put

$$x = y/\pi \quad \text{and} \quad \theta = \pi r/r_0.$$

Then (12) becomes

$$ru = \frac{2r_0^2u_0}{a\pi^2\sqrt{(\pi t)}}\int_0^\infty e^{-r_0^2y^2(4a^2\pi^2t)^{-1}}\,dy\sum_{n=1}^{n=\infty}(-1)^{n+1}n^{-1}\cos ny\sin n\theta \quad (13)$$

$$= \frac{r_0^2u_0}{a\pi^2\sqrt{(\pi t)}}\int_0^\infty e^{-r_0^2y^2(4a^2\pi^2t)^{-1}}\,dy\left\{\begin{array}{l} +\sum_{n=1}^{n=\infty}(-1)^{n+1}n^{-1}\sin n(y+\theta) \\ -\sum_{n=1}^{n=\infty}(-1)^{n+1}n^{-1}\sin n(y-\theta)\end{array}\right\}.$$

Again, for brevity, put

$$P = \frac{r_0^2u_0}{a\pi^2\sqrt{(\pi t)}},$$

$$p^2 = \frac{r_0^2}{4\pi^2a^2t};$$

and denote the difference of the two trigonometric series in the right-hand member of (13) by Q_1. Then, the expression to be evaluated is

$$ru = P\int_0^\infty e^{-p^2y^2}\,Q_1\,dy. \quad (14)$$

6. We must now determine the value of Q_1. For this purpose consider the well-known equation,

$$\frac{\frac{1}{2}(1-g^2)}{1-2g\cos\varphi+g^2} = \frac{1}{2} + g\cos\varphi + g^2\cos 2\varphi + g^3\cos 3\varphi + \dots\,.$$

In this, for φ, substitute $\pi + y + \varphi$. Then we have

$$\frac{1}{2} - \frac{\frac{1}{2}(1-g^2)}{1+2g\cos(y+\varphi)+g^2} = g\cos(y+\varphi) - g^2\cos 2(y+\varphi) + \dots\,.$$

Multiply this by $d\varphi$ and integrate between the limits $+\theta$ and $-\theta$. The result is

$$\begin{aligned}\tfrac{1}{2}\int_{-\theta}^{+\theta} d\varphi - \tfrac{1}{2}\int_{-\theta}^{+\theta}\frac{(1-g^2)\,d\varphi}{1+2g\cos(y+\varphi)+g^2} &= +\ g\sin\ (y+\theta) -\ g\sin\ (y-\theta)\\ &\quad -\tfrac{1}{2}g^2\sin 2(y+\theta) + \tfrac{1}{2}g^2\sin 2(y-\theta)\\ &\quad +\tfrac{1}{3}g^3\sin 3(y+\theta) - \tfrac{1}{3}g^3\sin 3(y-\theta)\\ &\quad - \,.\ .\ .\ .\ .\ .\ .\ .\ + \,.\ .\ .\ .\ .\ .\ .\ .\\ &= Q, \text{ say.} \qquad (15)\end{aligned}$$

Comparing the series in (13) and (14) with that in (15), it appears that $Q_1 = Q$ when $g = 1$. Hence equation (14) becomes

$$ru = \tfrac{1}{2}P\int_{-\theta}^{+\theta} d\varphi\int_{0}^{\infty} e^{-p^2y^2}\,dy - \tfrac{1}{2}P\int_{-\theta}^{+\theta}\frac{(1-g^2)\,d\varphi}{1+2g\cos(y+\varphi)+g^2}\int_{0}^{\infty} e^{-p^2y^2}\,dy. \qquad (16)$$

$$g = 1.$$

The first term in the second member of this equation presents no difficulty. Its value is

$$\frac{P\theta\sqrt{\pi}}{2p}\,.$$

But in the second term the element-function of the integral with respect to φ is zero when $g = 1$, except when $\cos(y+\varphi) = -1$. In this case the element-function becomes

$$\frac{1-g^2}{(1-g)^2} = \frac{1+g}{1-g} = \infty \quad \text{for} \quad g = 1.$$

The value of this integral with respect to φ is therefore the value of the single element $\frac{1+g}{1-g}\,d\varphi$ when $g = 1$. To find its value the most direct process appears to be the following*: —

Replace $\cos(y+\varphi)$ by its equivalent $1 - 2\sin^2\frac{1}{2}(y+\varphi)$. Then, after a little reduction, the integral becomes

$$\int_{-\theta}^{+\theta}\frac{\dfrac{1-g}{1+g}\sec^2\frac{1}{2}(y+\varphi)\,d(\frac{1}{2}\varphi)}{1+\left(\dfrac{1-g}{1+g}\right)^2\tan^2\frac{1}{2}(y+\varphi)}\,.$$

Now every element of this integral is zero for $g = 1$ except that for which

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* The mathematical reader will recognize in the integral to be evaluated one which Poisson and other writers have used in proving the law of development of a function in a series of periodic terms. The difficulty, if it be such, occurs in another form also in the proofs of that law given by Dirichlet and others. I am not certain that the evaluation given in the text will commend itself; but it seems to be more direct than, and quite as obvious as, Poisson's process.

$y + \varphi = (2n + 1)\pi$, n being any integer or zero; and the value of the exceptional element is dependent on θ only so far as it determines within what limits y can make $y + \varphi = (2n + 1)\pi$. We have, therefore, simply to consider what value the above expression has when $y + \varphi$ varies between $(2n + 1)\pi - i$ and $(2n + 1)\pi + i$, i being an infinitesimal. The indefinite integral is

$$\operatorname{arc\,tan}\left[\frac{1-g}{1+g}\tan\tfrac{1}{2}(y+\varphi)\right];$$

and since, when $(1 - g)$ is infinitesimal, $\frac{1-g}{1+g}\tan\frac{1}{2}(y + \varphi)$ varies from $-\infty$ to $+\infty$ as $y + \varphi$ varies over its range, the proper limits of integration are $-\infty$ and $+\infty$. Hence we conclude that

$$\frac{1}{2}\int_{-\theta}^{+\theta}\frac{(1-g^2)\,d\varphi}{1+2g\cos(y+\varphi)+g^2} = \left[\operatorname{arc\,tan}\left(\frac{1-g}{1+g}\tan\tfrac{1}{2}(y+\varphi)\right)\right]_{-\infty}^{+\infty} = \pi.$$

It remains to find what values of $y + \varphi$ will give $\cos(y + \varphi) = -1$, or what values of y will make $y + \varphi$ an odd multiple of π. Since φ may have any value between $-\theta$ and $+\theta$, the range of values of y, which make $y + \varphi$ an odd multiple of π, will be from

$$(2n+1)\pi - \theta \quad \text{to} \quad (2n+1)\pi + \theta,$$

n being any integer or zero. That is, y must have values lying between

$$\begin{array}{lcl} \pi - \theta & \text{and} & \pi + \theta, \\ 3\pi - \theta & \text{and} & 3\pi + \theta, \\ 5\pi - \theta & \text{and} & 5\pi + \theta, \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \end{array}$$

7. Recurring now to equation (16) and attending to the limits of y just derived we have

$$ru = \frac{P\theta\sqrt{\pi}}{2p} - P\pi\left(\int_{\pi-\theta}^{\pi+\theta} e^{-p^2y^2}\,dy + \int_{3\pi-\theta}^{3\pi+\theta} e^{-p^2y^2}\,dy + \ldots\right).$$

Restoring the values of P, p, and θ, and replacing py by z, there results

$$ru = ru_0 - \frac{2r_0u_0}{\sqrt{\pi}}\left(\int_{\frac{r_0-r}{2a\sqrt{t}}}^{\frac{r_0+r}{2a\sqrt{t}}} e^{-z^2}\,dz + \int_{\frac{3r_0-r}{2a\sqrt{t}}}^{\frac{3r_0+r}{2a\sqrt{t}}} e^{-z^2}\,dz + \ldots\right). \tag{17}$$

For brevity, let us put

$$\begin{aligned} m_0 &= r_0/(2a\sqrt{t}), \\ m &= r/(2a\sqrt{t}). \end{aligned} \tag{18}$$

Then observing that $\dfrac{2r_0u_0}{\sqrt{\pi}}\displaystyle\int_0^\infty e^{-z^2}\,dz = r_0u_0$,

we get the following equivalent forms from (17):—

$$ru = ru_0 - \frac{2r_0u_0}{\sqrt{\pi}}\left[\int_{m_0-m}^{m_0+m} e^{-z^2}\,dz + \int_{3m_0-m}^{3m_0+m} e^{-z^2}\,dz + \ldots\right], \tag{19}$$

$$ru = ru_0 - r_0u_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{m_0-m} e^{-z^2}\,dz - \frac{2}{\sqrt{\pi}}\int_{m_0+m}^{3m_0-m} e^{-z^2}\,dz - \ldots\right], \tag{20}$$

$$r(u_0-u) = \frac{2r_0u_0}{\sqrt{\pi}}\left[\int_{m_0-m}^{m_0+m} e^{-z^2}\,dz + \int_{3m_0-m}^{3m_0+m} e^{-z^2}\,dz + \ldots\right], \tag{21}$$

$$r(u_0-u) = r_0u_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{m_0-m} e^{-z^2}\,dz - \frac{2}{\sqrt{\pi}}\int_{m_0+m}^{3m_0-m} e^{-z^2}\,dz - \ldots\right]. \tag{22}$$

Equations (19) and (20) give the product of the radius r and the excess of temperature u of any shell of infinitesimal thickness at any time. Equations (21) and (22) give the product of the radius and the fall in temperature of the same shell during the time t.

8. It will now be well to determine whether equations (19) to (22) satisfy the conditions (1) to (5) and also to test their equivalence with (10).

Differentiating equation (20) or (22) with respect to t we find

$$\begin{aligned}\frac{\partial(ru)}{\partial t} &= \frac{2r_0u_0}{\sqrt{\pi}}e^{-(m_0-m)^2}\cdot\frac{\partial(m_0-m)}{\partial t}\\ &+ \frac{2r_0u_0}{\sqrt{\pi}}e^{-(3m_0-m)^2}\cdot\frac{\partial(3m_0-m)}{\partial t}\\ &- \frac{2r_0u_0}{\sqrt{\pi}}e^{-(m_0+m)^2}\cdot\frac{\partial(m_0+m)}{\partial t}\\ &+ \ldots\ldots\ldots\ldots\end{aligned}$$

But, from (18),

$$m_0 - m = \frac{r_0 - r}{2a}t^{-\frac{1}{2}},$$

$$m_0 + m = \frac{r_0 + r}{2a}t^{-\frac{1}{2}},$$

$$\ldots\ldots\ldots\ldots$$

Hence

$$\frac{\partial\,(m_0 - m)}{\partial t} = \frac{r_0 - r}{4a}\,t^{-\frac{3}{2}} = -\tfrac{1}{2}\,(m_0 - m)\,t^{-1},$$

$$\frac{\partial\,(m_0 + m)}{\partial t} = \frac{r_0 + r}{4a}\,t^{-\frac{3}{2}} = -\tfrac{1}{2}\,(m_0 + m)\,t^{-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,;$$

and

$$\frac{\partial\,(ru)}{\partial t} = -\frac{r_0 u_0}{t\sqrt{\pi}}\,\big[(m_0 - m)\,e^{-(m_0-m)^2} - (m_0 + m)\,e^{-(m_0+m)^2} + (3m_0 - m)\,e^{-(3m_0-m)^2} + \ldots\big]. \qquad (23)$$

Differentiating (20) with respect to r, there results,

$$\frac{\partial\,(ru)}{\partial r} = \frac{2r_0u_0}{\sqrt{\pi}}\left\{\begin{array}{l} e^{-(m_0-m)^2}\,.\,\dfrac{\partial\,(m_0 - m)}{\partial r} \\ +\,e^{-(3m_0-m)^2}\,.\,\dfrac{\partial\,(3m_0 - m)}{\partial r} \\ -\,e^{-(m_0+m)^2}\,.\,\dfrac{\partial\,(m_0 + m)}{\partial r} \\ +\,\ldots\ldots\ldots\ldots \end{array}\right\}.$$

By means of (18)

$$\frac{\partial\,(m_0 - m)}{\partial r} = -\frac{1}{2a\sqrt{t}},$$

$$\frac{\partial\,(3m_0 - m)}{\partial r} = -\frac{1}{2a\sqrt{t}},$$

$$\frac{\partial\,(m_0 + m)}{\partial r} = +\frac{1}{2a\sqrt{t}},$$

$$\cdots\cdots\cdots\cdots\cdots$$

Therefore

$$\frac{\partial\,(ru)}{\partial r} = u_0 - \frac{r_0u_0}{a\sqrt{(\pi t)}}\,\big[e^{-(m_0-m)^2} + e^{-(m_0+m)^2} + e^{-(3m_0-m)^2} + \ldots\big]. \qquad (24)$$

Differentiating the last equation again, we find

$$\frac{\partial^2\,(ru)}{\partial r^2} = -\frac{r_0u_0}{a^2t\sqrt{\pi}}\left\{\begin{array}{l} (m_0 - m)\,e^{-(m_0-m)^2} \\ -\,(m_0 + m)\,e^{-(m_0+m)^2} \\ +\,(3m_0 - m)\,e^{-(3m_0-m)^2} \\ -\,\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (25)$$

Dividing (23) by (25), we get $\dfrac{\partial\,(ru)}{\partial t} = a^2\,\dfrac{\partial^2\,(ru)}{\partial r^2}$,

which is equation (1). The equations (19) to (24) therefore fulfill the first requirement.

If we make $r = r_0$, equation (19) becomes

$$r_0 u = r_0 u_0 - \frac{2r_0 u_0}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\, dz$$

$$= r_0 u_0 - r_0 u_0 = 0,$$

and hence $u = 0$. Condition (2) is thus satisfied.

If we make $r = 0$, equation (20) gives

$$ru = 0 - r_0 u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\, dz\right) = 0,$$

as required by condition (3).

When $t = 0$, equation (20) becomes

$$ru = ru_0 - r_0 u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\, dz\right)$$

$$= ru_0,$$

or $$u = u_0,$$

which is in accordance with condition (4).

Finally, equations (19) to (22) should satisfy condition (5). But if we make $t = \infty$ in (21), the result is

$$r(u_0 - u) = \frac{2r_0 u_0}{\sqrt{\pi}} \left(\int_0^0 e^{-z^2}\, dz + \int_0^0 e^{-z^2}\, dz + \ldots\right),$$

which is ambiguous. The same ambiguity appertains to equation (22). This difficulty will appear also when we seek to determine the temperature at the centre of the sphere after an indefinitely great, or infinite, time. But it will be observed that if we can derive an expression from equations (19) to (22) for the temperature at the centre of the sphere, and show this expression to be zero when $t = \infty$, then those equations must satisfy (5).

If we make $r = 0$ in (20), it becomes

$$u = u_0 - r_0 u_0 \frac{1 - 1}{0}.$$

But the true value of this expression is readily found by differentiation to be

$$u = u_0 \left[1 - \frac{4m_0}{\sqrt{\pi}} \left(e^{-m_0^2} + e^{-9m_0^2} + e^{-25m_0^2} + \ldots\right)\right]. \qquad (26)$$

$$r = 0.$$

Now since $m_0 = r/2a\sqrt{t}$, the last equation assumes the indeterminate form when

$t = \infty$. But from the Eulerian integral of § 4 we have

$$e^{-n^2 m_0^2} = e^{-n^2 / m_0^{-2}} = \frac{1}{m_0\sqrt{\pi}} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2}\, dx \cos nx.$$

Making n successively equal to 1, 2, 3, . . . in this expression, and substituting the equivalents in (26), we get

$$u = u_0 \left[1 - \frac{4}{\pi} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2}\, dx \left(\cos x + \cos 3x + \cos 5x + \ldots\right)\right].$$

By reference to the series used in § 6 it is seen that when $g = 1$,

$$\frac{\frac{1}{4}(1 - g^2)}{1 - 2g\cos x + g^2} - \frac{\frac{1}{4}(1 - g^2)}{1 + 2g\cos x + g^2} = \cos x + \cos 3x + \cos 5x + \ldots .$$

Hence we have when $g = 1$,

$$u = u_0 \left[1 - \frac{4}{\pi} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2}\, dx \left(\frac{\frac{1}{4}(1 - g^2)}{1 - 2g\cos x + g^2} - \frac{\frac{1}{4}(1 - g^2)}{1 + 2g\cos x + g^2}\right)\right]. \tag{27}$$

When t is infinite, m_0 is infinitesimal, and we may for this value of t restrict the upper limit in (27) to an infinitesimal i, since for all greater values of x, $e^{-(\frac{1}{2}x / m_0)^2}$ vanishes. In this case, also, the second term under the sign of integration in (27) may be dropped, by reason of the infinitesimal factor $(1 - g^2)$. Therefore, when $t = \infty$ and $g = 1$, (27) becomes

$$u = u_0 \left[1 - \frac{1}{\pi} \int_{-i}^{+i} \frac{e^{-(\frac{1}{2}x / m_0)^2}(1 - g^2)\, d(\frac{1}{2}x)}{1 - 2g\cos x + g^2}\right]. \tag{28}$$

Since i may be assumed to be of a lower order than m_0, the value of the factor $e^{-(\frac{1}{2}x / m_0)^2}$ within the limits of integration is unity. The value of the integral then, as shown in § 6, is π. Therefore we conclude that when $r = 0$ and $t = \infty$,

$$u = u_0 \left[1 - (1 / \pi)\pi\right] = 0,$$

and hence that equations (19) to (22) satisfy condition (5).

9. Having established the complete equivalence of equations (19) to (22) and (10), we may now consider some practical features pertaining to their applications. In the first place it will be observed that the definite integrals in either of equations (19) to (22) form an extremely converging series for all but very great values of the time. So converging are these series that, for the case of the earth, the first term will suffice when t is as great 100,000,000,000 years. When t exceeds this time, recourse may be had to the original series (10), which then becomes highly converging also, as the following numerical examples will show.

10. If we confine the time to relatively small values, it will suffice to use the

first term of (19) to (20). If, further, we restrict r to values which do not differ sensibly from r_0, we may write (22) in the form

$$u_0 - u = u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{r_0 - r}{2a\sqrt{t}}} e^{-z^2} dz \right]. \tag{29}$$

This agrees with an expression which Sir William Thomson has used for the fall of temperature, in his paper, *On the Secular Cooling of the Earth.**

11. To illustrate the numerical application of the formulæ derived, it will be well to assume such a value of the time that for the earth both sets of equations, i. e. (10) and (11), and (19) to (24), and (26) may be applied.

As a first example let

$$r = \tfrac{1}{2} r_0 \quad \text{and} \quad a^2 (\pi / r_0)^2 t = 1.$$

The value of t required in the case of the earth to give this last equality is, as we have seen in §3, about one hundred thousand million years. With these data equation (10) gives

$$u = \frac{4u_0}{\pi} (e^{-1} - \tfrac{1}{3} e^{-9} + \ .\ .\ .\ ).$$

	log		No.
e^{-1}	9.5657055 — 10		+ 0.3678794
e^{-9}	6.09135 — 10	$\frac{1}{3}$	— 0.0000411
$(e^{-1} - \frac{1}{3} e^{-9} + \ .\ .\ .\ )$	9.5656570 — 10		0.3678383
4	0.6020600		
π	0.4971499		
u / u_0	9.6705671 — 10		0.4683463 ;

hence $u = 0.4683463\, u_0$.

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* Thomson and Tait's *Natural Philosophy*, Vol. I. Part II. Appendix (D).

The law of cooling used by Thomson in this paper is that which applies to the diffusion of temperature in an infinite solid on the supposition that at the beginning of the time the temperature had two different constant values on the two sides of an infinite plane. The formula which expresses this law is

$$v = v_0 + \frac{2V}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(\chi t)}}} e^{-z^2} dz,$$

in which v_0 is the half sum and V the half difference of the initial temperatures; x is the distance of any point whose temperature is v, from the initial plane, the sign of x being positive for the hotter side and negative for the cooler side of the plane; t is the time from the initial epoch and χ is the same as our a^2.

Now when $t = 0$ the formula gives $v = v_0 + V$, and hence the fall of temperature in any time t is

$$v_0 + V - \left[v_0 + \frac{2V}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(\chi t)}}} e^{-z^2} dz \right] = V \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{(\chi t)}}} e^{-z^2} dz \right].$$

This agrees with (29) if we make $V = u_0$ and $x = r_0 - r$.

Again, when $a^2(\pi / r_0)^2 t = 1$, $2a\sqrt{t} = 2r_0 / \pi$; and when $r = \frac{1}{2}r_0$, equations (18) give

$$\begin{aligned} m_0 &= \tfrac{1}{2}\pi, \\ m &= \tfrac{1}{4}\pi, \\ m_0 - m &= \tfrac{1}{4}\pi = 0.7853982, \\ m_0 + m &= \tfrac{3}{4}\pi = 2.3561945, \\ 3m_0 - m &= \tfrac{5}{4}\pi = 3.9269908, \\ 3m_0 + m &= \tfrac{7}{4}\pi = 5.4977871. \end{aligned}$$

From the tables of the integral $\int_0^z e^{-z^2} dz$ given in Oppolzer's *Lehrbuch zur Bahnbestimmung*, Vol. II. Table X, we find for the first two terms in (21)

$$\int_{\frac{7}{4}\pi}^{\frac{3}{4}\pi} e^{-z^2} dz = 0.2355828.78,$$

$$\int_{\frac{5}{4}\pi}^{\frac{7}{4}\pi} e^{-z^2} dz = 0.0000000.25.$$

The sum of these integrals is

$$\begin{array}{rll} & & \log \\ & 0.2355829 & 9.3721438 - 10, \\ & 4. & 0.6020600, \\ & \sqrt{\pi} & 0.2485749, \\ \dfrac{u_0 - u}{u_0}, & 0.5316538 & 9.7256289 - 10. \end{array}$$

Hence by (21) $$u_0 - u = 0.5316538\, u_0,$$

or $$u = 0.4683462\, u_0.$$

This agrees with the value derived above from (10) within one unit in the seventh place of decimals. It will be observed that the contribution of the second term in (21) is less than one unit in the seventh place. The first term of (21) would therefore suffice when the time is much greater than 100,000,000,000 years for all practical computations relative to the temperature of points within the earth

12. As a second illustration, let us compute the temperature at the centre of the sphere when m_0 of (18) and (26) has the value

$$m_0 = \frac{r_0}{2a\sqrt{t}} = 1.$$

This gives $$t = \frac{r_0^2}{4a^2},$$

which for the case of the earth gives $t = 273{,}000{,}000{,}000$ years. We also have for use in (11), when $m_0 = 1$,

$$a^2 \left(\frac{\pi}{r_0}\right) = \frac{\pi^2}{4}.$$

Hence the computation by (11) runs thus: —

	No.	log
1st term.	$\log e$	$9.6377843 - 10$
	π^2	0.9942997
	4	0.6020600
	$\log e^{(\frac{1}{2}\pi)^2}$	0.0300240
	$e^{(\frac{1}{2}\pi)^2}$	1.0715785
$+ 0.0848050$	$e^{-(\frac{1}{2}\pi)^2}$	$8.9284215 - 10.$
2d term.	$\log e^{(\frac{1}{2}\pi)^2}$	0.03002
	4	0.60206
	$\log e^{\pi^2}$	0.63208
	e^{π^2}	4.2863
$- 0.0000517$	$e^{-\pi^2}$	$5.7137_n - 10.$

The third and higher terms are insignificant. The sum of the first two is 0.0847533. Hence by (11)

$$u = 0.1695066\, u_0.$$

According to (26) the computation is as follows: —

		log
$+ 0.3678794$	e^{-1}	$9.5657055 - 10$
$+ 0.0001234$	e^{-9}	$6.09135 \quad - 10$
$+ 0.3680028$	$e^{-1} + e^{-9} + \ldots$	$9.5658511 - 10$
	4	0.6020600
	$\sqrt{\pi}$	0.2485749
0.8304935	$(4/\sqrt{\pi})(e^{-1} + e^{-9} + \ldots)$	$9.9193362 - 10.$

Therefore by (26)

$$u = u_0(1 - 0.8304935)$$
$$= 0.1695065\, u_0.$$

Thus the results derived from (11) and (26) for the temperature at the centre of the sphere agree within the limits of precision attainable with 7-place logarithms.